REMARK ON POSITIVE SOLUTIONS OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS *

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Abstract

For systems of retarded functional differential equations with unbounded delay and with finite memory sufficient conditions of existence of positive solutions on an interval of the form $[t_0, \infty)$ are derived.

1 Introduction

In this paper we give sufficient conditions for the existence of positive solutions (i.e. a solution with positive coordinates on a considered interval) for systems of retarded functional differential equations (RFDE's) with *unbounded delay* and with *finite memory.* At first let us give short explanation emphasized above terms.

Let us recall basic notions of RFDE's with unbounded delay but with finite memory. A function $p \in C[\mathbb{R} \times [-1, 0], \mathbb{R}]$ is called a p *-function* if it has the following properties [13, p. 8]:

(i) $p(t, 0) = t$.

(ii) $p(t, -1)$ is a nondecreasing function of t.

(iii) there exists a $\sigma \geq -\infty$ such that $p(t, \vartheta)$ is an increasing function for ϑ for each $t \in (\sigma, \infty)$. (Throughout the following text we suppose $t \in (\sigma, \infty)$.)

In the theory of RFDE's the symbol y_t, which expresses *"taking into account"*, the history of the process $y(t)$ considered, is used. With the aid of p - functions the symbol y_t is defined as follows:

Definition 1 ([13, p. 8]) Let $t_0 \in \mathbb{R}$, $A > 0$ and $y \in C([p(t_0, -1), t_0 + A), \mathbb{R}^n)$. For any $t \in [t_0, t_0 + A)$, we define

$$y_t(\vartheta) := y(p(t, \vartheta)), \ -1 \leq \vartheta \leq 0$$

and write

$$y_t \in \mathcal{C} := C[[-1, 0], \mathbb{R}^n].$$

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1.1 System with unbounded delay with finite memory

In this paper we investigate existence of positive solutions of the system

$$\dot{y}(t) = f(t, y_t) \tag{1}$$

where $f \in C([t_0, t_0 + A) \times \mathcal{C}, \mathbb{R}^n)$, $A > 0$, and y_t is defined in accordance with Definition 1. This system is called the system of *p-type retarded functional differential equations* (p-RFDE's) or a system *with unbounded delay with finite memory.*

Definition 2 The function

$$y \in C([p(t_0, -1), t_0 + A), \mathbb{R}^n) \cap C^1([t_0, t_0 + A), \mathbb{R}^n)$$

satisfying (1) on $[t_0, t_0 + A)$ is called a *solution of* (1) *on* $[\,p(t_0, -1), t_0 + A)$.

Suppose that Ω is an open subset of $\mathbb{R} \times \mathcal{C}$ and the function $f : \Omega \to \mathbb{R}^n$ is continuous. If $(t_0, \phi) \in \Omega$, then *there exists a solution* $y = y(t_0, \phi)$ of the system p-RFDE's (1) through (t_0, ϕ) (see [13, p. 25]). Moreover this *solution is unique* if $f(t, \phi)$ is locally Lipschitzian with respect to second argument ϕ ([13, p. 30]) and is *continuable* in the usual sense of extended existence if f is quasibounded ([13, p. 41]). Suppose that the solution $y = y(t_0, \phi)$ of p-RFDE's (1) through $(t_0, \phi) \in \Omega$, defined on $[t_0, A]$, is unique. Then the property of the *continuous dependence* holds too (see [13, p. 33]), i.e. for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $(s, \psi) \in \Omega$, $|s - t_0| < \delta$ and $\|\psi - \phi\| < \delta$ imply

$$\|y_t(s, \psi) - y_t(t_0, \phi)\| < \epsilon, \text{ for all } t \in [\zeta, A]$$

where $y(s, \psi)$ is the solution of the system p- RFDE's (1) through (s, ψ), $\zeta = \max\{s, t_0\}$ and $\|\cdot\|$ is the supremum norm in $\mathbb{R}^n$. Note that these results can be adapted easily for the case (which will be used in the sequel) when Ω has the form $\Omega = [\,p^*, \infty) \times \mathcal{C}$ where $p^* \in \mathbb{R}$.

1.2 Problem of existence of positive solutions

In this paper we are concerned with the problem of existence of *positive solutions* (i.e. problem of existence of solutions having all its coordinates positive on considered intervals) for nonlinear systems of RFDE's with unbounded delay but with finite memory. Let us cite some known results for retarded functional differential equations. Results in this direction are formulated in the book [12] and in the paper [1], too. Positive solutions in the critical case were studied e.g. in [4]–[10]. Some known scalar results concerning existence of positive solutions were extended for nonlinear systems of RFDE's with bounded retardation in [3] and for nonlinear systems of RFDE's with unbounded delay and with finite memory in [6].

2 Auxiliary lemma

With $\mathbb{R}^n_{\geq 0}$ ($\mathbb{R}^n_{>0}$) we denote the set of all component-wise nonnegative (positive) vectors v in $\mathbb{R}^n$, i. e., $v = (v_1, \dots, v_n) \in \mathbb{R}^n_{\geq 0}$ ($\mathbb{R}^n_{>0}$) if and only if $v_i \geq 0$ ($v_i > 0$) for $i = 1, \dots, n$. For $u, v \in \mathbb{R}^n$ we write $u \leq v$ if $v - u \in \mathbb{R}^n_{\geq 0}$; $u \ll v$ if $v - u \in \mathbb{R}^n_{>0}$ and $u < v$ if $u \leq v$ and $u \neq v$.

Let p^*, t^* be constants satisfying $p^* = p(t^*, -1)$ for a given p-function. Define vectors

$$\rho, \delta \in C([\,p^*, \infty), \mathbb{R}^n) \cap C^1([t^*, \infty), \mathbb{R}^n)$$

satisfying $\rho \ll \delta$ on $[\,p^*, \infty)$. Let us put $\Omega := [t^*, \infty) \times \mathcal{C}$ and

$$\omega := \{(t, y) : t \geq p^*,\ \rho(t) \ll y \ll \delta(t)\}.$$

Definition 3 A *system of initial functions* $\mathcal{S}_{\mathcal{E},\omega}$ with respect to nonempty sets $\mathcal{E}$ and ω where $\mathcal{E} \subset \overline{\omega}$ is defined as a continuous mapping $\nu : \mathcal{E} \to \mathcal{C}$ such that $a)$ and $b)$ in the following text hold:

$a)$ For each $z = (t, y) \in \mathcal{E} \cap \operatorname{int} \omega$ and $\vartheta \in [-1, 0] : (t + \vartheta, \nu(z)(p(t, \vartheta))) \in \omega$.

$b)$ For each $z = (t, y) \in \mathcal{E} \cap \partial\omega$ and $\vartheta \in [-1, 0) : (t + \vartheta, \nu(z)(p(t, \vartheta))) \in \omega$ and, moreover, $(t, \nu(z)(p(t, 0))) = z$.

We define as $\mathcal{S}^1_{\mathcal{E},\omega}$ a system of initial functions $\mathcal{S}_{\mathcal{E},\omega}$ if all functions $\nu(z)$, $z = (t, y) \in \mathcal{E}$ are continuously differentiable on $[-1, 0)$.

The next lemma deals with sufficient conditions for existence of solutions of the system (1), the graphs of which remain in the set ω. The proof of this lemma is based on the retract method and the Lyapunoff method and can be found in [6, Theorem 1]. Since this result will be used in the following, we modify slightly its original formulation underlying the necessary (for our purposes) fact that *every set* of initial functions contains at least *one initial function* generating solution with desired properties. This claim is a consequence of the proof of cited result.

Lemma 1 *Suppose $f \in C(\Omega, \mathbb{R}^n)$ is locally Lipschitzian with respect to the second argument, quasibounded and moreover:*

(i) For any $i = 1, \dots, p$ (with $p \in \{0, 1, \dots, n\}$), $t \geq t^*$ and $\pi \in C([p(t, -1), t], \mathbb{R}^n)$ such that $(\theta, \pi(\theta)) \in \omega$ for all $\theta \in [p(t, -1), t)$, $(t, \pi(t)) \in \partial\omega$ it follows $(t, \pi_t) \in \Omega$,

$$\delta_i'(t) < f_i(t, \pi_t) \ \text{when } \pi_i(t) = \delta_i(t) \tag{2}$$

and

$$\rho_i'(t) > f_i(t, \pi_t) \ \text{when } \pi_i(t) = \rho_i(t). \tag{3}$$

(ii) *For any $i = p+1, \dots, n$, $t \ge t^*$ and $\pi \in C([p(t,-1),t], \mathbb{R}^n)$ such that $(\theta, \pi(\theta)) \in \omega$ for all $\theta \in [p(t,-1),t)$, $(t, \pi(t)) \in \partial\omega$ it follows $(t, \pi_t) \in \Omega$,*

$$\delta_i'(t) > f_i(t, \pi_t) \ \text{when}\ \pi_i(t) = \delta_i(t) \tag{4}$$

and

$$\rho_i'(t) < f_i(t, \pi_t) \ \text{when}\ \pi_i(t) = \rho_i(t). \tag{5}$$

Then at every set of initial functions $\mathcal{S}_{\mathcal{E},\omega}$ with

$$\mathcal{E} := \{(t,y) : t = t^*, \rho(t) \le y \le \delta(t)\}$$

there exist at least one $\nu = \nu^* \in \mathcal{S}_{\mathcal{E},\omega}$ defined by a $z^* = (t^*, y^*) \in \mathcal{E} \cap \operatorname{int} \omega$ such that for corresponding solution $y(t^*, \nu^*(z^*))$ we have

$$(t, y(t^*, \nu^*(z^*))(t)) \in \omega \tag{6}$$

for every $t \ge p^*$.

3 Existence of Positive Solutions

Let

$$k := (k_1, \dots, k_n) \gg 0$$

be a constant vector and

$$\lambda(t) := (\lambda_1(t), \dots, \lambda_n(t))$$

denote a vector, defined and locally integrable on $[\,p^*, \infty)$. Define an auxiliary operator

$$T(k,\lambda)(t) := k \mathrm{e}^{\displaystyle\int_{p^*}^t \lambda(s)\mathrm{d}s} = \left(k_1 \mathrm{e}^{\displaystyle\int_{p^*}^t \lambda_1(s)\mathrm{d}s}, k_2 \mathrm{e}^{\displaystyle\int_{p^*}^t \lambda_2(s)\mathrm{d}s}, \dots, k_n \mathrm{e}^{\displaystyle\int_{p^*}^t \lambda_n(s)\mathrm{d}s} \right). \tag{7}$$

Let a constant vector $k \gg 0$ and a vector $\lambda(t)$ defined and locally integrable on $[\,p^*, \infty)$ are given. Then the operator T is well defined by (7). Define for every $i \in \{1, 2, \dots, n\}$ two type of subsets of the set $\mathcal{C}$:

$$\mathcal{T}^{\,i} := \left\{ \phi \in \mathcal{C} : 0 \ll \phi(\vartheta) \ll T(k,\lambda)_t(\vartheta),\ \vartheta \in [-1,0] \ \text{ except for }\ \phi_i(0) = k_i \mathrm{e}^{\displaystyle\int_{p^*}^t \lambda_i(s)\mathrm{d}s} \right\}$$

and

$$\mathcal{T}_i := \{\phi \in \mathcal{C} : 0 \ll \phi(\vartheta) \ll T(k,\lambda)_t(\vartheta),\ \vartheta \in [-1,0] \text{ except for } \phi_i(0) = 0\}\,.$$

Theorem 1 *Suppose $f \in C(\Omega, \mathbb{R}^n)$ is locally Lipschitzian with respect to the second argument and quasibounded. Let a constant vector $k \gg 0$ and a vector $\lambda(t)$ defined and locally integrable on $[\,p^*, \infty)$ are given. If, moreover, inequalities*

$$\mu_i \lambda_i(t) > \frac{\mu_i}{k_i}\, \mathrm{e}^{-\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \cdot f_i\,(t, \phi) \tag{8}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathcal{T}^{\,i}$ and inequalities

$$\mu_i f_i(t, \phi) > 0 \tag{9}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathcal{T}_i$, where $\mu_i = -1$ for $i = 1, \ldots, p$ and $\mu_i = 1$ for $i = p+1, \ldots, n$, then there exists a positive solution $y = y(t)$ on $[\,p^*, \infty)$ of the system p-RFDE's (1)*.*

Proof. We will employ Lemma 1. Put $\rho(t) := 0$, $\delta(t) := T(k, \lambda)(t)$. Let us suppose $i \in \{1, \ldots, p\}$. It is easy to conclude that inequality (2) is equivalent to

$$\delta_i'(t) < f_i(t, \phi) \ \text{ when } \ \phi \in \mathcal{T}^{\,i} \tag{10}$$

if the function π is changed by the function $\phi \in \mathcal{T}^{\,i}$ and inequality (3) is equivalent to

$$\rho_i'(t) > f_i(t, \phi) \ \text{ when } \ \phi \in \mathcal{T}_i \tag{11}$$

if the function π is changed by the function $\phi \in \mathcal{T}_i$. Similarly, for $i \in \{p+1, \ldots, n\}$ we conclude that inequality (5) is equivalent to

$$\delta_i'(t) > f_i(t, \phi) \ \text{ when } \ \phi \in \mathcal{T}^{\,i} \tag{12}$$

if the function π is changed by the function $\phi \in \mathcal{T}^{\,i}$ and inequality (4) is equivalent to

$$\rho_i'(t) < f_i(t, \phi) \ \text{ when } \ \phi \in \mathcal{T}_i \tag{13}$$

if the function π is changed by the function $\phi \in \mathcal{T}_i$. Let us verify that above inequalities are valid. For $t \geq t^*$ and $i \in \{1, \dots, p\}$ (i.e. $\mu_i = -1$) we get:

$$f_i(t,\phi) - \delta_i'(t) = \mu_i(\delta_i'(t) - f_i(t,\phi)) = \mu_i \left(k_i \lambda_i(t) \mathrm{e}^{\displaystyle\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} - f_i(t,\phi) \right) =$$

$$k_i \mathrm{e}^{\displaystyle\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \left(\mu_i \lambda_i(t) - \frac{\mu_i}{k_i} \mathrm{e}^{\displaystyle -\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} f_i(t,\phi) \right) >$$

$$[\text{in view of } (8)] > k_i \mathrm{e}^{\displaystyle\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} (\mu_i \lambda_i(t) - \mu_i \lambda_i(t)) = 0.$$

Similarly, for $t \geq t^*$ and $i \in \{p+1, \dots, n\}$ (i.e. $\mu_i = 1$) we get:

$$\delta_i'(t) - f_i(t,\phi) = \mu_i(\delta_i'(t) - f_i(t,\phi)) = \mu_i \left(k_i \lambda_i(t) \mathrm{e}^{\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} - f_i(t,\phi) \right) =$$

$$k_i \mathrm{e}^{\displaystyle\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \left(\mu_i \lambda_i(t) - \frac{\mu_i}{k_i} \mathrm{e}^{\displaystyle -\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} f_i(t,\phi) \right) >$$

$$[\text{in view of } (8)] > k_i \mathrm{e}^{\displaystyle\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} (\mu_i \lambda_i(t) - \mu_i \lambda_i(t)) = 0.$$

Therefore inequalities (10), (12) hold. Inequalities (11), (13) are valid, too since, due to (9)

$$\rho_i'(t) - f_i(t,\phi) = \mu_i f_i(t,\phi) > 0, \quad \text{if } i = 1, 2, \dots, p \ \text{ (i.e. } \mu_i = -1)$$

and

$$f_i(t,\phi) - \rho_i'(t) = \mu_i f_i(t,\phi) > 0 \quad \text{if } i = p+1, p+2, \dots, n \ \text{ (i.e. } \mu_i = 1).$$

All conditions of Lemma 1 are satisfied. From its conclusion we immediately get the desired statement. Theorem 1 is proved. □

Remark 1 Let us underline that if Theorem 1 hold, then indicated positive solution $y = y(t)$ satisfies on $[p^*, \infty]$ inequalities

$$0 \ll y(t) \ll \delta(t)$$

with corresponding given δ.

3.1 A nonlinear example

The following example demonstrates that results can be successfully applied to nonlinear systems. Let us show that the system

$$\begin{aligned} y_1'(t) &= -\tfrac{1}{2}\left[y_1^4(t^{1/2}) + y_1^2(t)\cdot y_2(t)\right], \\ y_2'(t) &= y_2(t) - y_1(t)\cdot y_2(t^{1/2})\cdot y_3(t), \\ y_3'(t) &= y_1^2(t^{1/2})\cdot y_3^2(t^{1/2}) \end{aligned} \tag{14}$$

has a positive solution on interval $[2,\infty)$. Define

$$p(t,\vartheta) := t + (t-\sqrt{t})\vartheta, \ \ \vartheta \in [-1,0].$$

Then the system (14) can be rewritten as

$$\begin{aligned} y_1'(t) &= f_1(t,y_t) := && -\tfrac{1}{2}\left[y_1^4(p(t,-1)) + y_1^2(p(t,0))\cdot y_2(p(t,0))\right], \\ y_2'(t) &= f_2(t,y_t) := && y_2(p(t,0)) - y_1(p(t,0)\cdot y_2(p(t,-1))\cdot y_3(p(t,0)), \\ y_3'(t) &= f_3(t,y_t) := && y_1^2(p(t,-1))\cdot y_3^2(p(t,-1)). \end{aligned}$$

Let us verify that Theorem 1 can be used. For it we put:

$$\begin{aligned} & p^* = 2 = p(t^*,-1), \\ & t^* = 4, \\ & k \ = (k_1,k_2,k_3) = (1/4, 1, 1/2), \\ & \lambda \ = (\lambda_1,\lambda_2,\lambda_3) = (-1/t, 0, 1/t), \\ & \mu_1 = \mu_2 = -1 \\ & \mu_3 = 1. \end{aligned}$$

Then

$$T(k,\lambda)(t)) := k e^{\int_2^t \lambda(s)\mathrm{d}s} = \left(\frac{1}{4}\cdot \mathrm{e}^{-\int_2^t \mathrm{d}s/s}, 1, \frac{1}{2}\cdot \mathrm{e}^{\int_2^t \mathrm{d}s/s}\right) = \left(\frac{1}{2t}, 1, \frac{t}{4}\right).$$

Let us verify inequalities (8) and (9). If $i = 1$ and $\phi \in \mathcal{T}^1$ then

$$\frac{\mu_1}{k_1}\mathrm{e}^{-\int_{p^*}^t \lambda_1(s)\mathrm{d}s} \cdot f_1(t,\phi) =$$

$$-2t\cdot f_1(t,\phi) < t\cdot\left[\left(\frac{1}{2\sqrt{t}}\right)^4 + \left(\frac{1}{2t}\right)^2\right] = \frac{3}{8t} < \frac{1}{t} = \mu_1\lambda_1(t),$$

if $i = 2$ and $\phi \in \mathcal{T}^2$ then

$$\frac{\mu_2}{k_2} \mathrm{e}^{-\int_{p^*}^{t} \lambda_2(s)\mathrm{d}s} \cdot f_2\,(t, \phi) = -\frac{2}{t} \cdot f_2\,(t, \phi) =$$

$$-\frac{2}{t} \cdot \left[1 - \phi_2(-1) \cdot \frac{1}{2t} \cdot \frac{t}{4}\right] < \frac{2}{t} \cdot \left[-1 + \frac{1}{8}\right] = -\frac{7}{4t} < 0 = \mu_2 \lambda_2(t)$$

and if $i = 3$ and $\phi \in \mathcal{T}^3$ then

$$\frac{\mu_3}{k_3} \mathrm{e}^{-\int_{p^*}^{t} \lambda_3(s)\mathrm{d}s} \cdot f_3\,(t, \phi) =$$

$$\frac{4}{t} \cdot f_3\,(t, \phi) < \frac{4}{t} \cdot \left(\frac{1}{2\sqrt{t}}\right)^2 \cdot \left(\frac{\sqrt{t}}{4}\right)^2 = \frac{1}{16t} < \frac{1}{t} = \mu_3 \lambda_3(t)$$

and inequalities (8) on interval $[4, \infty)$ hold.

Inequalities (9) hold on interval $[4, \infty)$ since if $i = 1$ and $\phi \in \mathcal{T}_1$ then

$$\frac{\mu_1}{k_1} \cdot f_1\,(t, \phi) = -4 f_1\,(t, \phi) = 2\left[\phi_1^4(-1) + \phi_1^2(0) \cdot \phi_2(0)\right] > 0,$$

if $i = 2$ and $\phi \in \mathcal{T}_2$ then

$$\frac{\mu_2}{k_2} \cdot f_2\,(t, \phi) = -f_2\,(t, \phi) =$$

$$-\left[\phi_2(0) - \phi_1(0) \cdot \phi_2(-1) \cdot \phi_3(0)\right] = \phi_1(0) \cdot \phi_2(-1) \cdot \phi_3(0) > 0$$

and if $i = 3$ and $\phi \in \mathcal{T}_3$ then

$$\frac{\mu_3}{k_3} \cdot f_3\,(t, \phi) = \frac{1}{2} f_3\,(t, \phi) = \frac{1}{2}\left[\phi_1^2(-1) \cdot \phi_3^2(-1)\right] > 0.$$

All conditions of Theorem 1 are valid. Therefore a positive solution

$$y = y(t) = (y_1(t), y_2(t), y_3(t)),$$

of the system (14) exists on $[2, \infty)$. Taking into account Remark 1 we conclude that on the interval considered inequalities

$$\begin{aligned} 0 &< y_1(t) < 1/2t\,, \\ 0 &< y_2(t) < 1, \\ 0 &< y_3(t) < t/4 \end{aligned}$$

hold.

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