INITIAL DATA GENERATING SOLUTIONS OF A GENERAL QUADRATIC DIFFERENCE EQUATIONS WITH PRESCRIBED ASYMPTOTIC BEHAVIOUR

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Abstract. The discrete quadratic equation
\[ u(k + 1) = \alpha(k)(u(k))^2 + \beta(k)u(k) - \gamma(k), \]
with variable \( k \in N(a) \) is considered. A result concerning asymptotic behavior of its solutions for \( k \to \infty \) is formulated and corresponding two-sided inequalities for them are given. In the paper we treat the problem how to determine initial data generating solutions with such asymptotic behavior. It is shown that in some cases it is possible to get corresponding initial data with the aid of limits of two number auxiliary sequences.

1 Introduction

In this paper we consider the discrete quadratic equation
\[ u(k + 1) = \alpha(k)(u(k))^2 + \beta(k)u(k) - \gamma(k), \quad (1) \]
with variable \( k \in N(a) \). Throughout this paper we suppose \( \alpha, \beta, \gamma : N(a) \to \mathbb{R}^+ \) for all \( k \in N(a) \), where \( \mathbb{R}^+ : = (0, +\infty) \). Define \( N^* : = \{0\} \cup N \). We will show that under certain conditions the equation (1) has at least one solution defined on \( N(a) \) with prescribed asymptotic behavior. In the paper we treat the problem how to determine initial data \((a, u^*)\) generating solutions with such a behavior. It is shown that in some cases it is possible to get corresponding initial data (i.e. value \( u^* \)) with the aid of limits of two special auxiliary sequences.

1.1 Preliminaries

Put \( \Delta u(k) := u(k + 1) - u(k) \). Let us consider a scalar discrete equation containing one difference
\[ \Delta u(k) = f(k, u(k)) \quad (2) \]
with a one-valued function \( f : N(a) \times \mathbb{R} \to \mathbb{R} \) and \( a \in N \). Together with discrete equation (2) we consider an initial problem. It is posed as follows: for a given \( s \in N^* \) we are seeking the solution \( u = u(k) \) of (2) satisfying the initial condition
\[ u(a + s) = u_s \in \mathbb{R} \quad (3) \]
with a prescribed constant $u_s$. Let us recall that the solution of initial problem (2), (3) is defined as an infinite sequence of numbers $\{u^k\}_0^\infty$ with $u^k = u(a + s + k)$, i.e.

$$u^0 = u_s = u(a + s), u^1 = u(a + s + 1), \ldots, u^n = u(a + s + n), \ldots$$

such that for any $k \in N(a + s)$ the equality (2) holds. Let us note that the existence and uniqueness of the solution of the initial problem (2), (3) is a consequence of properties of the function $f$. Suppose moreover that $f$ is continuous with respect to the second argument. Then the initial problem (2), (3) depends continuously on its initial data.

Let $b$, $c$ be real functions defined on $N(a)$ such that $b(k) < c(k)$ for every $k \in N(a)$. Let us define a set $\omega \subset N(a) \times \mathbb{R}$ as

$$\omega := \{(k, u) : k \in N(a), u \in \omega(k)\}$$

where

$$\omega(k) := \{(u) : b(k) < u < c(k)\}.$$

Let us define a closure of the set $\omega$ as

$$\overline{\omega} := \{(k, u) : k \in N(a), u \in \overline{\omega}(k)\},$$

with

$$\overline{\omega}(k) := \{(u) : b(k) \leq u \leq c(k)\}.$$

The following theorem, concerning an asymptotic behaviour of solutions of the equation (2) is a partial case of Theorem 1 from [2] (see [3] as well).

**Theorem 1.** Let $b$, $c$ be real functions defined on $N(a)$ such that $b(k) < c(k)$ for every $k \in N(a)$. Let $f : \overline{\omega} \to \mathbb{R}$ be continuous with respect to the second argument. If, moreover, for every $k \in N(a)$

$$f(k, b(k)) - b(k + 1) + b(k) < 0$$

and

$$f(k, c(k)) - c(k + 1) + c(k) > 0$$

then there exists an initial condition

$$u^*(a) = u^* \in \omega(a) \quad (4)$$

such that the corresponding solution $u = u^*(k), k \in N(a)$ of equation (2) satisfies the relation

$$u^*(k) \in \omega(k) \quad (5)$$

for every $k \in N(a)$.
1.2 Application of Theorem 1 to Equation (1)

The quadratic equation (1) is a partial case of equation (2) with the right-hand side

\[ f(k, u) := \alpha(k)(u(k))^2 + \beta(k)u(k) - \gamma(k) - u(k). \]

In the sequel we will suppose that \( b, c \) are real functions defined on \( N(a) \) such that \( 0 < b(k) < c(k) \) for every \( k \in N(a) \). Let us reformulate Theorem 1 with respect to equation (1).

**Theorem 2.** If for every \( k \in N(a) \)

\[ \alpha(k)(b(k))^2 + \beta(k)b(k) - \gamma(k) < b(k + 1) \quad (6) \]

and

\[ \alpha(k)(c(k))^2 + \beta(k)c(k) - \gamma(k) > c(k + 1) \quad (7) \]

then there exists an initial condition (4) such that the corresponding solution \( u = u^*(k), \, k \in N(a) \) of equation (1) satisfies the relation (5) for every \( k \in N(a) \).

Let us underline that Theorem 2 (as well as Theorem 1) states only that there exists an initial condition (4) generating solution \( u = u^*(k), \, k \in N(a) \) of corresponding equation satisfying (5) for every \( k \in N(a) \) without giving any concrete determination of the corresponding initial data \( u^* \) itself. In this contribution we fill this gap particularly in the case of equation (1). A lot of investigations concern to various asymptotic problems for discrete equations (e.g. in [1, 4]–[6]). The above formulated problem was not a topic of investigation yet.

2 Auxiliary Operators and Sequences

The below introduced operators will be defined on \( \mathbb{R}^+ \). Let us define with the aid of the right-hand side of equation (1) an operator \( T_k \):

\[ T_k(u) := \alpha(k)u^2 + \beta(k)u - \gamma(k) \]

for every \( k \in N(a) \),

\[ T_k(u) = \alpha(k) \left( u^2 + \frac{\beta(k)u}{\alpha(k)} \right) - \gamma(k) = \]

\[ = \alpha(k) \left[ \left( u + \frac{\beta(k)}{2\alpha(k)} \right)^2 - \frac{(\beta(k))^2}{4(\alpha(k))^2} \right] - \gamma(k) \]
and corresponding inverse operator $T_k^{-1}$ (the existence of which follows from conditions given above):

$$T_k^{-1}(u) := \sqrt{\frac{u + (\beta(k))^2 + \gamma(k)}{2\alpha(k)}} - \frac{\beta(k)}{2\alpha(k)}$$

Next we define for every $s \in \mathbb{N}^*$ operators $L_{a+i}$, $i = 0, 1, \ldots, s$ and inverse operators $L_{a+i}^{-1}$, $i = 0, 1, \ldots, s$ as:

$$L_a(u) = u,$$
$$L_{a+1}(u) := T_a(u),$$
$$L_{a+2}(u) := T_{a+1}(T_a(u)),$$
$$\ldots$$
$$L_{a+s}(u) := T_{a+s-1}(T_{a+s-2}(...(T_{a+1}(T_a(u))))\ldots)$$

and

$$L_a^{-1}(u) := u,$$
$$L_{a+1}^{-1}(u) := T_a^{-1}(u),$$
$$L_{a+2}^{-1}(u) := T_{a+1}^{-1}(T_a^{-1}(u)),$$
$$\ldots$$
$$L_{a+s}^{-1}(u) := T_{a+s-1}^{-1}(...(T_{a+1}^{-1}(T_{a+s-2}^{-1}(T_{a+s-1}^{-1}(u))))\ldots).$$

Now we give some properties of introduced operators.

**Lemma 1.** Operators $T_k$, $T_k^{-1}$, $L_{a+i}$, $L_{a+i}^{-1}$, $i = 0, 1, \ldots, s$, $s \in \mathbb{N}^*$ are increasing on $\mathbb{R}^+$ for every fixed $k \in N(a)$ and $i \in \{0, 1, \ldots, s\}$.

**Lemma 2.** For every $u \in \mathbb{R}^+$ and $s \in \mathbb{N}^*$:

$$L_{a+s}^{-1}(T_{a+s}^{-1}(u)) = L_{a+s+1}^{-1}(u). \quad (8)$$

**Lemma 3.** Let the inequalities (6), (7) be valid for every $k \in N(a)$. Then the sequence $\{u_{cs}\}_{s=0}^{\infty}$ with

$$u_{cs} := L_{a+s}^{-1}(c(a + s)), \quad s \in \mathbb{N}^*, \quad (9)$$

is a decreasing convergent sequence, and the sequence $\{u_{bs}\}_{s=0}^{\infty}$ with

$$u_{bs} := L_{a+s}^{-1}(b(a + s)), \quad s \in \mathbb{N}^*, \quad (10)$$

is an increasing convergent sequence. Moreover $u_{cs} > u_{bs}$ holds for every $s \in \mathbb{N}^*$ and for corresponding limits

$$c^* = \lim_{s \to \infty} u_{cs}, \quad b^* = \lim_{s \to \infty} u_{bs}, \quad (11)$$
the inequality $c^* \geq b^*$ holds.

Let us formulate an elementary property of solutions.

**Lemma 4.** Solutions $u(k), U(k), k \in N(a)$ of two initial conditions for equation (1):
\[ u(a) = \alpha, \quad \text{and} \quad U(a) = \beta \]
with $0 < \alpha < \beta$ satisfy the inequalities
\[ u(k) < U(k) \]
for every $k \in N(a)$.

**Lemma 5.** Let the inequalities (6), (7) be valid for every $k \in N(a)$. Then

a) The solution $u = u^*_{cs}(k), k \in N(a)$ of the initial condition
\[ u^*_{cs}(a) = u_{cs}, \quad s \in N^* \]
for the equation (1) satisfies the relations
\[ u^*_{cs}(k) \in \omega(k) \]
for every $k = a, a+1, \ldots, a+s-1$ and
\[ u^*_{cs}(a+s) = c(a+s). \]
Moreover,
\[ u^*_{cs+1}(k) < u^*_{cs}(k) \quad \text{if} \quad k = a, a+1, \ldots, a+s. \]

b) The solution $u^*_{bs}(k), k \in N(a)$ of the initial condition
\[ u^*_{bs}(a) = u_{bs}, \quad s \in N^* \]
for the equation (1) satisfies the relations
\[ u^*_{bs}(k) \in \omega(k) \]
for every $k = a, a+1, \ldots, a+s-1$ and
\[ u^*_{bs}(a+s) = b(a+s). \]
Moreover,
\[ u^*_{bs+1}(k) > u^*_{bs}(k) \quad \text{if} \quad k = a, a+1, \ldots, a+s. \]
3 Main Results

In this final part we give results concerning the existence of at least one solution of the problem (1), (5).

**Theorem 3.** [Main Result] Let the inequalities (6), (7) be valid for every \( k \in N(a) \). Then every initial condition (4) with \( u^* \in [b^*, c^*] \), where \( b^* \) and \( c^* \) are defined by (11), determines a solution of equation (1) satisfying relation (5).

**Consequence 1.** If Theorem 3 holds then a solution of the problem (1), (4) satisfying relation (5) is defined by

\[
    u(a + s) = L_{a+s}(u^*), \quad s \in N^*
\]

with \( u^* \in [b^*, c^*] \).

**Theorem 4.** Let the inequalities (6), (7) be valid for every \( k \in N(a) \). Then the initial condition \( u(a) = u^\nabla \) with \( u^\nabla \in [b(a), c(a)] \setminus [b^*, c^*] \) generate a solution \( u = u^\nabla(k), \ k \in N(a) \) of equation (1) not satisfying relation (5) for all \( k \in N(a) \).

**Corollary 2.** Let the inequalities (6), (7) be valid for every \( k \in N(a) \). Then a solution \( u = u(k), \ k \in N(a) \) of equation (1) satisfies relation (5) for every \( k \in N(a) \) if and only if \( u(a) \in [b^*, c^*] \).

**Corollary 3.** Let the inequalities (6), (7) be valid for every \( k \in N(a) \). Let, moreover, \( b^* = c^* \). Then the equation (1) has a unique solution \( u = u^*(k), \ k \in N(a) \) satisfying for every \( k \in N(a) \) relation (5). This solution is determined by initial data \( u^*(a) = u^* = b^* \).

Let us find sufficient conditions for the case \( b^* = c^* \). Let the inequalities (6), (7) be valid for every \( k \in N(a) \). Denote

\[
    \Delta(s) = u_{cs} - u_{bs}, \quad s = 0, 1, \ldots
\]

Then the length of the interval \([b^*, c^*]\) can be estimated (due to the monotonicity of sequences \( \{u_{cs}\}_{s=0}^{\infty}, \{u_{bs}\}_{s=0}^{\infty} \)) as

\[
    0 \leq c^* - b^* < \Delta(s), \quad s = 0, 1, \ldots
\]

From the definition of the expressions \( u_{cs}, u_{bs} \) we see that

\[
    \Delta(s) = u_{cs} - u_{bs} = [\text{due to (9) and (10)}] = L_{a+s}^{-1}(c(a + s)) - L_{a+s}^{-1}(b(a + s)).
\]

Then

\[
    0 < u_{cs} - c^* < \Delta(s), \quad s \in N
\]

\[
    0 < b^* - u_{bs} < \Delta(s), \quad s \in N.
\]
Theorem 5. Let the inequalities (6), (7) be valid for every $k \in N(a)$. Then $b^* = c^*$ if
\[
\lim_{s \to \infty} \Delta(s) = 0.
\]

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