Numerical range and numerical radius (An introduction)

K. F. Žitný, J. Kozánek*

Abstract

In early studies of Hilbert spaces (by Hilbert, Hellinger, Toeplitz, and others) the object of chief interest were quadratic forms. Nowadays, quadratic questions about a linear continuous operator are questions about its numerical range - [1], [3], [5]. Students may find in this paper a motivation to go on to a deeper understanding of the properties of linear operators and matrices as well as some numerical methods how to determine their numerical ranges. It is rather surprising that the proof of the so called Elliptical Range Theorem for the matrices of order 2 is tedious - [4].

We deal with the Banach algebra $\mathbb{B}(H)$ of linear continuous operator on a complex Hilbert space H. The numerical range of an operator $T \in \mathbb{B}(H)$ is the subset of the complex field $\mathbb{C}$, given by

$$\mathcal{V}(T) = \{(Tx|x) \mid x \in H,\ ||x|| = 1\}.$$

The next fundamental result is known as the Toeplitz-Hausdorff theorem.

Theorem The numerical range $\mathcal{V}(T)$ of $T \in \mathbb{B}(H)$ is convex.

This theorem has many proofs, a recent one is due to C.K. Li „C-Numerical Ranges and C-Numerical Radii", Linear and Multilinear Algebra (1994), 37, 51-82.

The numerical radius of $T \in \mathbb{B}(H)$ is given by

$$v(T) = \sup\{|\lambda| \mid \lambda \in \mathcal{V}(T)\}.$$

Obviously, $v(T^*) = v(T)$ for every $T \in \mathbb{B}(H)$, and for any vector $x \in H$, we have

$$|(Tx|x)| \leq v(T) \cdot ||x||^2.$$

Lemma For $T \in \mathbb{B}(H)$, $\frac{1}{2}||T|| \leq v(T)$, and $v(T^2) \leq v(T)^2$.

*Institute of Thermomechanics AS CR, Dolejškova 5, 182 00 Prague 8, e-mail: kozanek@it.cas.cz

In the following H denotes a Hermitian space. Let $\mathcal{L}(H)$ denotes the algebra of linear operators on H. By $\mathcal{K}(\mathbb{C})$ we denote the metric space of nonempty compact subsets of $\mathbb{C}$ endowed with the Hausdorff metric Δ. If $K_1, K_2 \in \mathcal{K}(\mathbb{C})$, then

$$\Delta(K_1, K_2) = \max[\sup\{dist(\lambda, K_1) \,|\, \lambda \in K_2\}, \sup\{dist(\lambda, K_2) \,|\, \lambda \in K_1\}]$$

where $dist(\lambda, K_j) = \inf\{|\lambda - \mu| \,|\, \mu \in K_j\}$, $j = 1, 2$. The most important facts about $\mathcal{V}(T)$ are the following

Theorem

a) If $T \in \mathcal{L}(H)$, then $\mathcal{V}(T)$ is a compact, convex subset of $\mathbb{C}$, and the numerical radius is attained.

b) $\sigma(T) \subset \mathcal{V}(T)$ for every $T \in \mathcal{L}(H)$.

c) $\mathcal{V}(T + S) \subset \mathcal{V}(T) + \mathcal{V}(S)$, $T, S \in \mathcal{L}(H)$.

d) $\Delta(\mathcal{V}(T), \mathcal{V}(S)) \leq ||T - S||$, $T, S \in \mathcal{L}(H)$.

This research was partially supported by the Academy of Sciences of the Czech Republic under contract No. IBS2076301.

References

[1] Halmos P. R.: A Hilbert Space Problem Book, Second Edition, Springer, 1982

[2] Marcus M.: Matrices and MATLAB: A Tutorial, Prentice Hall,1993

[3] Horn R. A., Johnson Ch. R.: Topics in matrix analysis, Cambridge, 1994

[4] Godunov S. K.: Sovremennyje aspekty linejnoj algebry, NK, Novosibirsk, 1997

[5] Gustafson K. E., Rao D. K. M.: Numerical Range (The Field of Values of Linear Operators and Matrices), Springer, 1997