Approximation of Periodical Functions

Mgr. Marie Tomšová

We often meet with the following problem in technical pracsice. A function $f(x)$ which is rather complied is given on interval I and it is necessary to recompense it by a more simple function whose values can be simply enumerated and which is "near enough" to $f(x)$ on interval I. We can suppose for such a function for example polynomial

$$P_n(x) = a_0 + a_1 x + \ . \ . \ . \ + a_n x^n \qquad (1)$$

ore trigonometrical polynomial

$$T_n(x) = a_0/2 + (a_1 \cos x + b_1 \sin x) + \ . \ . \ . \ + (a_n \cos nx + b_n \sin nx). \qquad (2)$$

We suppose that the function $f(x)$ is defined on the interval $[a, b]$ and $G_n(x)$ is either algebraical or trigonometrical polynomial. We suppose choice of $n + 1$ different points in the interval $[a, b]$:

$$x_0,\ x_1,\ x_2,\ . \ . \ . \ , x_n . \qquad (3)$$

It is to find a function $G_n(x)$ satisfying the following properties: $f(x_i) = G_n(x_i)$ for $i = 0, 1, 2, \ . \ . \ . \ , n$. Function $G_n(x)$ is called interpolating function and the points x_i bundles of interpolation. We put $x_0 = a$ and $x_a = b$. The process of replacement of the function $f(x)$ by function $G_n(x)$ outside of the interval $[a, b]$ is called extrapolation. The most applied interpolating polynomials are: Lagrange, Newton, Gauss, Stierling, Bessel and other polynomials.
When we know about a (discrete) set of values of a function $f(x)$ which is in the investigated interval periodical or arose as an elementary function of the quotient of two polynomials then it is possible to suppose their extrapolations for arbitrary small number ε as very good in the sense of (4) .
As we have introduced, one class of functions which can be interpolated well is the class of periodical functions. This fact incites following question. Does there exist any possibility to verify by a mathematical method the existence of periodicity or at least of a hidden periodicity in the sequence of given values? We show one of the statistical methods which works with trigonometrical polynomials. Therefore we calculate aproximations of Fourier coefficients for the aproximation by polynomial (2).
We suppose in our considerations that the studied functions are time functions independent on t, their values at the points $t_1, t_2, \ . \ . \ . \ , t_n$ generate time series. It is well known that time series can be described by a system of sine and cosine waves with various amplitudes and frequencies in full generality in the form of trigonometrical polynomial.

$$f(t) = A_0 + \sum_{j=1}^{H} a_j \sin \omega_j t + \sum_{j=1}^{H} b_j \sin \omega_j t \qquad (5)$$

where $H = \frac{n}{2}$ for even n and $H = \frac{n-1}{2}$ for odd n, $\omega_j = \frac{2\pi j}{n}$ for $j = 1, \ . \ . \ . \ , H$ is the j^{th}-frequency and A_0 , a_j , b_j are coefficients, which will be estimated later. We introduce the notion of Fourier's period $\tau_j = \frac{2\pi}{\omega_j} = \frac{n}{j}$, which has the dimension of the time t which is

different from the physical definition of frequency, where the quotient $\frac{\omega}{2\pi}$ is supposed as the number of cycles per time period.
Some of Fourier's periods will be supposed as more significant and others no . Formally we suppose that the first m (where $m < H$) of these Fourier period τ_j are significant. The other $H - m$ are supposed as insignificant and we do not work with them hereafter. The equation (5) transforms into the form

$$f(t) = A_0 + \sum_{j=1}^{m} a_j \sin \omega_j t + \sum_{j=1}^{m} b_j \sin \omega_j t \qquad (6)$$

where the estimation of A_0 is $\overline{A_0} = \frac{1}{n}\sum_{i=1}^{n} f(t_i)$ and the estimations of a_j are

$\overline{a_j} = \frac{2}{n}\sum_{i=1}^{n} f(t_i) \sin \omega_j t_i$ and similarly b_j are estimated by

$\overline{b_j} = \frac{2}{n}\sum_{i=1}^{n} f(t_i) \cos \omega_j t_i$, where $j = 1, 2, \ldots m$.

For the establishment of existence of significant periodical component in time period we are in need of determination of the variance of estimated values $\overline{f(t_i)}$. The following formula can be proved:

$$\text{var } \overline{f(t_i)} = \frac{1}{2}\sum_{j=1}^{m} \left(\overline{a_j}^2 + \overline{b_j}^2 \right) , \qquad (7)$$

The determination of this variance will be instrumental for the construction and the analysis of periodogram . An important role in determining significant hidden periods has so called index of determination I^2 which is defined by the formula

$$I^2 = \frac{\text{var}\, \overline{f(t_i)}}{\text{var}\, f(t_i)} \qquad (8)$$

For identification of periodicity besides visual methods, objective methods of the analysis of time periods are often used. Especially auto-correlation functions, coherent functions but also the analysis of periodogram.

REFERENCES

[1] *Anděl, J.* : Matematická statistika. Praha, Matfyzpress 1993.
[2] *Cramér, H.*: Mathematical Methods of Statistics. Princeton, Princeton University Press 1956.
[3] *David, H.* : Order Statistics. New York, J. Wiley 1966.
[4] *Likeš, J –Laga, J.*: Základní statistické tabulky. SNTL Praha 1978.
[5] *Rousas, George G.:* A Course in Mathematical Statistics. Academic Press New York, 1997.
[6] *Zapletal, J.* :Úvod do analýzy ekonomických časových řad. PC-DIR, Brno 2000.

Author's addresses:
Department of Mathematics FEEC
Technical university Brno.
Technická 8, 616 00 Brno
E mail tomsova@feec.vutbr.cz