Parallelogram identity, its application and generalization

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This contribution deals with lesser-known, but interesting and useful parallelogram identity and issues from the monography Kosmák, L., Potůček, R.: *Metrické prostory*, Academia, Praha 2004. The students of the Faculty of the Military Technologies of the University of Defence in Brno are acquainted with the basic notions of a linear algebra and of metric spaces, such as a linear space, a scalar product, a norm, a distance and an angle in the course of the mandatory subject "Mathematics" in the bachelor study and furthermore in the course of the facultative advanced subject „Linear algebra" in the master study.

During the master study of the subject "Linear algebra" the students are firstly acquainted with the the axiomatic definition of a linear space, which is accompanied by a number of examples and counter-examples. Further the notions of the linear combination, the linear dependence and independence, the base of a linear space, its dimension and the notion of the linear subspace are mentioned. The following notions are a scalar product, and a Euclidean space. Then the basic notions – a Euclidean norm and a metric are introduced. It follows an orthogonal system of vectors including a Gram-Schmidt orthogonalization and a orthogonal projection of a vector on the linear subspace is stated. These notions are succeeded by the notions of a linear transformation and then by a kernel, a rank and a defect of this transformation. The subject "Linear algebra" is concluded by the explanation of eigenvalues and eigenvectors of a matrix represented a linear transformation, including the matrix diagonalization constructed from the corresponding eigenvalues.

The above explication is convenient, according to my practice and own experience, to complete with several theorems which can to the students approximate some relations between the basic notions, such as following theorem:

Cauchy-Bunyakovski-Schwarz Theorem. If x,y are elements of a Euclidean space, then it holds inequality (so called CBS-inequality)

$$\left|\langle x, y\rangle\right| \le \|x\| \cdot \|y\|.$$

Further it follows the definition of a norm induced by the scalar product:

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

It is convenient to show that not all norms are induced by the scalar product (an example see in the contribution).

Very interesting theorem, which deals with norms of vectors and which has an illustrative geometrically sense (see the contribution), is so called

Parallelogram identity. For arbitrary vectors x,y of a Euclidean space we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Useful and important is the following theorem, which expresses the scalar product by means of a norm:

Polarization identity. For arbitrary vectors x,y of a Euclidean space we have

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

The parallelogram identity is a sufficient condition in the

Jordan-von Neumann theorem. If the parallelogram identity holds in a normed vector space, then a norm of this space is induced by the scalar product.
In the contribution is also presented an elegant proof of this theorem.

Finally, it is put the generalization of the parallelogram identity – also called

Tetragonal identity. For arbitrary vectors x,y,z,w of a normed vector space it holds

$$\|x-y\|^2 + \|y-z\|^2 + \|z-w\|^2 + \|w-x\|^2 = \|x-z\|^2 + \|y-w\|^2 + \|x+z-y-w\|^2.$$

I suppose, that the theorems above, some of which have a concrete geometrically sense, appropriately complete the standard reading of the subject “Linear algebra”.

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