Differential equations of orthogonal grids

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Abstract

In this contribution we summarize and extend well-known examples of orthogonal trajectories in elementary mathematics. Orthogonality is required and useful property in many technical applications.

Consider the system of curves $F(x, y, c) = 0$ in the plane, where $c \in \mathbf{R}$, and let $\Phi(x, y, y') = 0$ be their differential equation. Hence $\Phi(x, y, -\frac{1}{y'}) = 0$ is the differential equation of their orthogonal trajectories $G(x, y, k) = 0$, where $k \in \mathbf{R}$.

Consider two orthogonal systems of curves in the plane $\varrho = f_1(\varphi, c)$, $\varrho = f_2(\varphi, c)$ in polar coordinates, where $c \in \mathbf{R}$. The condition of orthogonality of these systems is

$$\frac{\mathrm{d}f_1}{\mathrm{d}\varphi}.\frac{\mathrm{d}f_2}{\mathrm{d}\varphi} = -\varrho^2.$$

Consider the autonomous system of differential equations in the plane

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(x(t), y(t)\right), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g\left(x(t), y(t)\right), \tag{1}$$

where f a g are continuous functions in a domain $\Omega \subseteq \mathbf{R}^2$. The solution of the system (1) is $x = x(t)$, $y = y(t)$. The graph of solution of the system (1) is a curve in $\Omega \times \mathbf{R}$. The perpendicular projection of the graph of solution to the domain Ω is a curve given in the parametrical form $x = \varphi(t)$, $y = \psi(t)$, where $t \in J \subseteq \mathbf{R}$, called a trajectory of the system (1).

Solving the system $x' = f(x, y)$, $y' = g(x, y)$ is the same as solving differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$ in the domain $G_1 = \{(x, y) \in \Omega : f(x, y) \neq 0\}$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x,y)}{g(x,y)}$ in the domain $G_2 = \{(x, y) \in \Omega : g(x, y) \neq 0\}$. It can be expressed in a symmetrical form $f(x, y)\mathrm{d}y - g(x, y)\mathrm{d}x = 0$.

Using the complex value $z = x + \mathrm{i}y$ we can write the autonomous system $x' = f(x, y)$, $y' = g(x, y)$ like

$$z' = F(z), \tag{2}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}t}$ and $F(z) = f(x, y) + \mathrm{i}g(x, y)$ is a complex function.

Consider two autonomous systems $x' = f(x, y)$, $y' = g(x, y)$, and $x' = -g(x, y)$, $y' = f(x, y)$. The trajectories of both autonomous systems create an orthogonal grid in $\Omega \subseteq \mathbf{R}^2$.

If we multiply the function $F = f + \mathrm{i}g$ by the imaginary unit i, we obtain the function $\mathrm{i}F = -g + \mathrm{i}f$.

The theory of stationary points of linear autonomous systems is well-known.

Denote $f_1 = \frac{\partial f(x_0,y_0)}{\partial x}$, $f_2 = \frac{\partial f(x_0,y_0)}{\partial y}$, $g_1 = \frac{\partial g(x_0,y_0)}{\partial x}$, $g_2 = \frac{\partial g(x_0,y_0)}{\partial y}$. Suppose that $f(x, y)$, $g(x, y)$ are continuous functions having continuous partial derivatives of second order in the neighbourhood of the point $[x_0, y_0]$, $f_1 g_2 - f_2 g_1 \neq 0$, and $f(x_0, y_0) = g(x_0, y_0) = 0$. Then the point $[x_0, y_0]$ is an isolated stationary point of the system $x' = f(x, y)$, $y' = g(x, y)$. Likewise if the point $[0, 0]$ is an isolated stationary point of the linear autonomous system $x' = f_1 x + f_2 y$, $y' = g_1 x + g_2 y$, and his type is node, focus or saddle point, the type of stationary point $[x_0, y_0]$ of the system $x' = f(x, y)$, $y' = g(x, y)$ is the same. If the point $[0, 0]$ is isolated stationary point of the linear autonomous system $x' = f_1 x + f_2 y$, $y' = g_1 x + g_2 y$, and his type is center, the type of stationary point $[x_0, y_0]$ of the system $x' = f(x, y)$, $y' = g(x, y)$ is rotation point or focus.

Now we investigate several special functions $F(z)$ in the equation $z' = F(z)$:

If $F = f(x) + \mathrm{i}g(y)$, then we obtain the autonomous system $x' = f(x)$, $y' = g(y)$. We can solve it as two differential equations $\frac{\mathrm{d}x}{f(x)} = \mathrm{d}t$, $\frac{\mathrm{d}y}{g(y)} = \mathrm{d}t$, where $f \neq 0$, $g \neq 0$, respectively. We can replace this autonomous system by a differential equation $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(x)}{g(y)}$ for $f \neq 0$, or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(y)}{f(x)}$ for $g \neq 0$. Hence we have an equation $\frac{y'}{g(y)} = \frac{1}{f(x)}$, where $f \neq 0$, $g \neq 0$.

If $F = f(y) + \mathrm{i}g(x)$, we obtain the autonomous system $x' = f(y)$, $y' = g(x)$. We solve it like an equation $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{f(y)}{g(x)}$ for $g \neq 0$, or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{f(y)}$ for $f \neq 0$. Hence we have equation $f(y)y' = g(x)$ or $g(x)x' = f(y)$.

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If $F = af(x, y) + \mathrm{i}bf(x, y)$, where $a, b \in \mathbf{R}$, then we obtain the autonomous system $x' = af(x, y)$, $y' = bf(x, y)$. We obtain $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{a}{b}$ for $b \neq 0$, or $\frac{dy}{dx} = \frac{b}{a}$ for $a \neq 0$. This is an equation $bx' = a$, $ay' = b$, respectively, and its solution is $ay - bx = c$, where $c \in \mathbf{R}$.

Especially, for $a = 0$ we obtain $x' = 0$, hence $x(t) = \text{const.}$, $y' = bf(y, c)$, where $c \in \mathbf{R}$, which gives an equation $\frac{dy}{bf(y,c)} = dt$, whose solutions are parallel to yt-plane in $\Omega \times \mathbf{R}$.

Similarly for $b = 0$ we obtain $y' = 0$, hence $y(t) = \text{const.}$, $x' = af(x, c)$, where $c \in \mathbf{R}$, which gives an equation $\frac{dx}{af(x,c)} = dt$, whose solutions are parallel to xt-plane in $\Omega \times \mathbf{R}$.

The aim of this contribution is to construct the collection of examples of differential equations of orthogonal trajectories depending on parameters. In the following examples we investigate orthogonal trajectories of systems of curves like solutions of convenient differential equations. In many cases this method is very useful. We will formulate several geometrical problems, we will solve them using analytical methods and their solutions will be interpreted geometrically.

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