SOME PROBLEMS IN APPLIED TOPOLOGY

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We will discuss two rings of problems, represented by two author's recent papers.

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In Problem 527, stated in the well-known book Open Problems in Topology, J. Lawson and M. Mislove ask, for which DCPO's the Scott topology has a basis consisting of open filters and for which DCPO's the topology generated by the Scott open filters is T_0. We introduce a notion of Hofmann-Mislove poset, for which we can partially answer the question.

Theorem 1.1. *Let $(X, \leq)$ be a Hofmann-Mislove DCPO, $P \subseteq X$ its generalized spectrum. The following conditions are equivalent:*

(i) *The Scott topology on X has a base of Scott open filters.*

(ii) *For every up-filtered compact saturated set $K \subseteq X$ there is a family $\{K_i\}_{i \in I}$ of up-filtered compact sets such that $K = \bigcap_{i \in I} K_i$ and $K_i = \downarrow (P \cap K_i)$.*

(iii) *Every up-filtered compact saturated set in X is an intersection of saturations of up-filtered compact subsets of the generalized spectrum of X.*

Corollary 1.1. *Let $(X, \leq)$ be a frame. Then the Scott topology on X is the de Groot dual ω^d of the upper interval topology ω. Moreover, the Scott topology on X has a base of Scott open filters if and only if the saturations of compacts subsets of the spectrum of X form a closed base of ω^d.*

Theorem 1.2. *Let $(X, \leq)$ be a Hofmann-Mislove DCPO, $P \subseteq X$ its generalized spectrum. The following conditions are equivalent:*

(i) *The topology on X generated by the Scott open filters is T_0.*

(ii) *For every $x, y \in X$ there exists up-filtered compact $L \subseteq P$ such that one of the points x, y is and the other is not contained in the saturation of L.*

(iii) *For every $x, y \in X$, $x \neq y$ there is a prime element $p \in X$ such that $x \leq p$, $y \nleq p$ or $x \nleq p$, $y \leq p$.*

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Recall that a topology is said to be the de Groot dual of another topology, if it has a closed base consisting of all compact saturated sets. In the second ring of problems, we extend and investigate the notion of de Groot dual also to topological systems and locales. We compare the behavior of this extension with the properties of the de Groot dual for topological spaces, and study the similarities as well as the differences. Recall that the author already solved (in the positive) another question of J. Lawson and M. Mislove, known as Problem 540, whether the sequence of iterated de Groot duals of a topological space is finite. Hence, in this contribution we state and study a natural question: For which locales is the extended de Groot dual of finite order?

We say that the topological system $(X', P, \models)$ is a *dualization* of $(X, A, \vdash)$ *represented* by the frame P if there exists a morphism (f, e) of topological systems from $(X, (\tau_X(A))^d, \in)$ to $(X', P, \models)$ such that $e : P \to (\tau_X(A))^d$ is a frame epimorphism. The dualization $(X', P, \models)$ of $(X, A, \vdash)$ is *strict* if f is a bijection. The basic properties of the defined concepts are described by the following theorem.

Theorem 2.1. *Let $(X, A, \vdash)$ be a topological system, P be a frame. The following conditions are equivalent:*

(i) *P represents a dualization of $(X, A, \vdash)$.*
(ii) *P represents a strict dualization of $(X, A, \vdash)$.*
(iii) *There exists a frame epimorphism $e : P \to (\tau_X(A))^d$.*

Let $\mathbf{2}^A$ be the set of all mappings of A to the Sepiński frame $\mathbf{2}$, where we put $y \leq z$ if and only if $y(a) \leq z(a)$ for every $a \in A$. Obviously, $\mathbf{2}^A$ is a frame isomorphic to the power set 2^A ordered by inclusion. Let $P \subseteq \mathbf{2}^A$. Let us denote $[P] = \bigcap \big\{S |\, P \subseteq S \subseteq \mathbf{2}^A,\ S$ is closed under all joins and finite meets in $\mathbf{2}^A \big\}$. Let $y : A \to \mathbf{2}$ be a mapping and $(X, A, \vdash)$ a topological system. We say that y is a *compact function* in $(X, A, \vdash)$ if there exists a compact set $K \subseteq X$ such that $y = w_K$. If $K = \{p\}$, we write $w_K = w_{\{p\}} = w_p$. The set containing all compact functions in $(X, A, \vdash)$ we denote by $\mathcal{C}(X, A)$. Further, we say that $x \in X$ is *independent* on $y \in [\mathcal{C}(X, A)]$ and write $x \models y$ if there is some $a \in A$ such that $y(a) = \mathcal{T}$ and $x \nvdash a$.

Theorem 2.2. *Let $(X, A, \vdash)$ be a topological system. Then $(X, [\mathcal{C}(X, A)], \models)$ is a topological system which is a strict dualization of $(X, A, \vdash)$.*

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