Straight Lines in Threedimensional Space and the Ultrahyperbolic Equation

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1. The Inverse Problem of the Calculus of Variations

We deal with the variational integral

$$L = \int_a^b f(x, y(x), z(x), y'(x), z'(x))\mathrm{d}x \;\to\; \min\ .$$

It is well-known that the extremals satisfy the Euler-Lagrange system

$$f_y = \frac{\mathrm{d}}{\mathrm{d}x} f_{y'}\ , \quad f_z = \frac{\mathrm{d}}{\mathrm{d}x} f_{z'}\ .$$

It follows that the straight lines $y = Ax + B$, $z = Cx + D$ are identical with the extremals if and only if the identities

$$\mathcal{E}:\ f_y = f_{y'x} + f_{y'y}y' + f_{y'z}z'\ ,\ f_z = f_{z'x} + f_{z'y}y' + f_{z'z}z'$$

hold true. These equations may be regarded as a system of partial differential equations for the unknown function $f = f(x, y, z, y', z')$ of five independent variables. Every solution f satisfying the regularity property $\Delta = f_{y'y'}f_{z'z'} - (f_{y'z'})^2 \neq 0$ implies that the extremals of the integral L are just the straight lines in the threedimensional space.

2. Link to the Ultrahyperbolic Equation

Identities $\mathcal{E}$ clearly imply

$$\begin{aligned} f_{yz'} &= f_{y'zz'} + f_{y'yz'}y' + f_{y'zz'}z' + f_{y'z}\ , \\ f_{zy'} &= f_{z'zy'} + f_{z'yy'}y' + f_{z'y} + f_{z'zy'}z'\ , \end{aligned}$$

whence the ultrahyperbolic equation $\mathcal{U}\ :\ f_{yz'} = f_{zy'}$ follows. We have a solution f depending moreover on a parameter x.

3. The Main Result

A somewhat unexpectedly, if $\bar{f} = \bar{f}(y, z, y', z')$ is a solution of $\mathcal{U}$ such that $\bar{\Delta} = \bar{f}_{y'y'}\bar{f}_{z'z'} - (\bar{f}_{y'z'})^2 \neq 0$, then there exists a solution $f = f(x, y, z, y', z')$ of $\mathcal{E}$ such that $\mathcal{J}:\ f(c, y, z, y', z') = \bar{f}(y, z, y', z')$ for an appropriate constant c. The formula for this particular function reads as follows

$$f(x, y, z, y', z') = -\left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right) g(x, y, z, y', z')$$

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where

$$g(x,y,z,y',z') = g_0 + (c-x)\int_0^1 \Big(\bar f_{y'}(\ldots)(y'-v_0) + \bar f_{z'}(\ldots)(z'-w_0)\Big)\,\mathrm{d}t\,,$$

$$\begin{aligned}(\ldots) \;=\; & (y+(c-x)(v_0+t(y'-v_0))\,,\; z+(c-x)(w_0+t(z'-w_0))\,,\\ & v_0+t(y'-v_0)\,,\; w_0+t(z'-w_0))\,,\end{aligned}$$

with arbitrary functions $g_0 = g_0(x,y,z)$, $v_0 = v_0(x,y,z)$, $w_0 = w_0(x,y,z)$. Every function f satisfying these conditions is a solution of $\mathcal{E}$ and the additional condition $g_0(c,y,z)=0$ ensures moreover the property $\mathcal{J}$. The result is derived by the use of the Poincaré-Cartan forms and this method can be adapted to resolve the General Inverse Problem where the extremals, not necessarily the straight lines, are given in advance and we search for the kernel function f in the integral L. Alternative approaches by Douglas and Anderson / Thompson are similar but more complex and also lead to the ultrahyperbolic equation. However, the connections are much more complicated and the explicit formulae cannot be easily obtained.

4. Examples

One can explicitly obtain many particular solutions f of $\mathcal{E}$ and consequently the solutions $\bar f$ of $\mathcal{U}$, e. g., the exponential solutions (rather easy), the polynomial solutions (with interesting interrelation to the Bessel functions), some examples resulting in the so-called Busemann projective metrics and their generalization (related to the ancient Menelaos Theorem), in particular the non-Euclidean hyperbolic geometry.

References

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