Velocity approximation in finite-element method for density-driven porous media flow*

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We deal with the numerical solution of groundwater flow and solute transport problems, related to important application in mining technologies and environmental protection. We introduce a numerical scheme for porous-media fluid flow with variable density, based on the finite element method (FEM) adapted to typical geometry of groundwater problems. The terms variable-density or density-driven flow denote the coupling of flow and transport problem, whereas the solute concentration distribution influences the velocity field by means of inhomogeneous solution density [2]. It requires suitable technique for "compatible" expression of discrete values of velocity and density.

The equation of variable-density flow is

$$-\nabla \cdot (K(\nabla h + \varrho_r \nabla z)) = q \quad \text{or} \quad -\nabla \cdot (K \nabla h) = q + \frac{\partial}{\partial z}(\varrho_r \nabla z)\,, \quad (1)$$

where h is the piezometric head $h = \frac{p}{\varrho_0 g} + z$ (freshwater equivalent), K is the hydraulic conductivity, q is the rate of sources/sinks, p is the pressure, ϱ_0 is the freshwater density, ϱ is the solution density, $\varrho_r = \frac{\varrho - \varrho_0}{\varrho_0}$ is relative density and g is the gravity acceleration. Besides h, the Darcy velocity $\boldsymbol{u} = K(\nabla h + \varrho_r \nabla z)$ is the unknown of interest, as the input value for the solute transport problem.

The mesh is derived form the unstructured 2D triangulation, with set of nodes in a vertical line associated with a node of the triangulation.

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The mesh consist of layers (in general not horizontal), with vertically connected nodes. In the sense of volumetric objects, the mesh consists of trilateral prisms, ordered to columns.

Since the mesh prisms are much larger in the horizontal direction than in the vertical direction, the formulation the FE base functions on prims is not natural for this case. We show an alternative numerical solution of the flow problem by means of combining the FEM with linear base functions on 2D triangle mesh and finite differences in the vertical direction (1D columns of mesh nodes). This approach generalize the idea of quasi-3D groundwater models [3]. The scheme can be also formulated in terms of finite volume method, providing the mass balance of discrete fluxes on the dual mesh (volumes surrounding the nodes of the original mesh) [1], important for subsequent solution of the solute transport problem and coupling in the density-driven flow problems. The solute transport problem is solved by cell-centered finite volume method on the dual mesh, i.e with values of concentrations associated with nodes.

We show how to generalize the relation for discrete fluxes along edges between nodes $u_{ij} = A_{ij}(h_i - h_j)$, where the A_{ij} is the stiffness matrix component corresponding to the two nodes and $h_{i,j}$ are the unknown piezometric heads. The additional contribution to the flux resulting from variable density is expressed from the discrete values of density in nodes using the stiffness matrix component as transmissibilities. The discrete representation of the density-dependent right-hand side term in the flow equation is derived as the difference of discrete fluxes, which keeps the conservative flux approximation with respect to the dual mesh.

References

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