Compulsory asymptotic behavior of solutions of two-dimensional systems of difference equations

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In this contribution we consider a system of two difference equations

$$\begin{array}{rcl} \Delta u_1(k) & = & f_1(k, u_1(k), u_2(k)), \\ \Delta u_2(k) & = & f_2(k, u_1(k), u_2(k)) \end{array} \tag{1}$$

where $k \in N(a) := \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed, $\Delta u_i(k) = u_i(k+1) - u_i(k)$, $i = 1, 2$, and $f_1, f_2 : N(a) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are functions that are continuous with respect to their last two arguments, i.e. $f_i(k, u_1, u_2)$, $i = 1, 2$, is continuous with respect to u_1 and u_2 for every fixed $k \in N(a)$.

Let us recall that the solution of system (1) is defined as an infinite sequence of number vectors

$$\{(u_1(k), u_2(k)\}_{k=a}^{\infty}$$

such that for any $k \in N(a)$ equalities (1) holds.

The existence and uniqueness of the solution of initial problem (1), (2) with

$$(u_1(a), u_2(a)) = (u_1^a, u_2^a) \in \mathbb{R} \times \mathbb{R} \tag{2}$$

on $N(a)$ is obvious.

The sequence

$$\{(k, u_1(k), u_2(k))\}, \ k \in N(a),$$

is called the graph of the solution $u = (u_1, u_2) = (u_1(k), u_2(k))$ for $k \in N(a)$ of initial problem (1), (2).

We are looking for sufficient conditions for the right-hand side of system (1) that guarantee the existence of at least one solution $u = u^*(k) = (u_1^*(k), u_2^*(k)),\ k \in N(a)$, the graph of which stays in a prescribed set Ω.
The considered set Ω is defined as

$$\Omega := \{(k, u_1, u_2) : k \in N(a), \, (k, u_1, u_2) \in \Omega(k)\}$$

where $\Omega(k) \subset \{(k, u_1, u_2) : (u_1, u_2) \in \mathbb{R}^2\}$ is for every $k \in N(a)$ a nonempty open bounded and connected set.
We will deal with sets $\Omega(k)$ that can be written in the form

$$\Omega(k) = \{(k, u_1, u_2) : b_i(k) < u_i < c_i(k), i = 1, 2\} \tag{3}$$

where $b_i, c_i : N(a) \to \mathbb{R}$, $i = 1, 2$ are auxiliary functions such that $b_i(k) < c_i(k)$ for every $k \in N(a)$.
These sets are called polyfacial sets. Obviously, the boundary of a polyfacial set consists of four parts, each of them being characterized by one of the conditions $u_1 = b_1(k), u_1 = c_1(k), u_2 = b_2(k)$ or $u_2 = c_2(k)$.

Theorem 1 *Let $b_i(k), c_i(k), b_i(k) < c_i(k), i = 1, 2$ be real functions defined on $N(a)$ and let $f_i : N(a) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, 2$ be functions that are continuous with respect to their last two arguments.*
Suppose that for all points $(k, u_1, u_2) \in \partial\Omega$ conditions (4)–(7) hold.

$$u_1 = b_1(k) \;\Rightarrow\; f_1(k, b_1(k), u_2) < b_1(k+1) - b_1(k), \tag{4}$$

$$u_1 = c_1(k) \;\Rightarrow\; f_1(k, c_1(k), u_2) > c_1(k+1) - c_1(k), \tag{5}$$

$$u_2 = b_2(k) \;\Rightarrow\; b_2(k+1) - b_2(k) < f_2(k, u_1, b_2(k)) < c_2(k+1) - b_2(k), \tag{6}$$

$$u_2 = c_2(k) \;\Rightarrow\; b_2(k+1) - c_2(k) < f_2(k, u_1, c_2(k)) < c_2(k+1) - c_2(k). \tag{7}$$

Let, moreover, function $F(w) = w + f_2(k, u_1, w)$ be monotone for every fixed $(k, u_1) \in \{(k, u_1) : k \in N(a), b_1(k) \le u_1 \le c_1(k)\}$ on the interval $b_2(k) \le w \le c_2(k)$. Then there exists a solution $u = (u_1^*(k), u_2^*(k))$ of system (1) satisfying inequalities

$$\begin{aligned} b_1(k) &< u_1^*(k) < c_1(k), \\ b_2(k) &< u_2^*(k) < c_2(k) \end{aligned} \tag{8}$$

for every $k \in N(a)$.

The proof of Theorem 1 is based on the so called retract type technique. The assumption that there exists no solution with the desired properties leads to the conclusion that a closed interval can be continuously mapped onto its (both) end points. This gives a contrary with the known fact that a closed n-dimensional ball cannot be continuously mapped onto its boundary.

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References

[1] Agarwal R. P., *Differential Equations and Inequalities, Theory, Methods, and Applications,* Marcel Dekker, Inc., 2nd ed., 2000.

[2] Boichuk A. & Růžičková M., Solutions of nonlinear difference equations bounded on the whole line, *Folia Facultatis Scientarium Naturalium Universitatis Masarykiana Brunensis*, **13** (2002), 45 – 60.

[3] Diblík J., Anti-Lyapunov method for systems of discrete equations, *Nonlinear Anal., Theory Methods Appl.,* **57** (2004), 1043–1057.

[4] Diblík J., Asymptotic behaviour of solutions of discrete equations, *Funct. Differ. Equ.,* **11** (2004), 37–48.

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